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LARGE AMPLITUDE VIBRATION OF CIRCULAR PLATES INCLUDING TRANSVERSE SHEAR AND ROTATORY INERTIA

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Abstract—This study is an analytical investigation of large amplitude flexural vibration of clamped circular plates with stress-free and immovable edges. The effects of transverse shear deformation and rotatory inertia are included in the governing equations. Solutions are formulated on the basis of Galerkin's method and the Runge–Kutta numerical procedure. An excellent agreement is found between the present results and those reported earlier for nonlinear static and dynamic cases. Numerical results indicate that the effects of transverse shear deformation and rotatory inertia are significant in the nonlinear dynamic analysis of circular plates, particularly for moderately thick plates.

1. INTRODUCTION

The large amplitude vibrations of plates have been discussed by a number of investigators[1]. Using Galerkin's method, Yamaki[2] has investigated the nonlinear transverse vibration of simply supported and clamped isotropic circular plates with stress-free and immovable edges. Nonlinear oscillations of a rectilinearly orthotropic circular plate with a clamped stress-free edge have been considered by Nowinski[3]. Axisymmetric nonlinear vibrations of thin circular plates have been also reported by Bulkelay[4], Srinivasan[5] and Kung and Pao[6]. However, in all these analyses the effects of the transverse shear deformation and rotatory inertia have been neglected. Recently, Kanaka Raju and Venkateswara Rao[7] have studied the effects of geometric nonlinearity, shear deformation and rotatory inertia on axisymmetric vibrations of circular plates. The finite element method has been used and results have been reported for simply supported and clamped isotropic circular plates with immovable boundaries. It may be noticed that the individual effects of either the transverse shear deformation or the rotatory inertia on the dynamic behavior of circular plates cannot be investigated in the analysis.

This paper is analytically concerned with nonlinear flexural vibrations of clamped moderately thick circular isotropic plates with immovable and stress-free edges. The effect of transverse shear deformation and rotatory inertia is included in this work in such a way that the individual or the combined effect can be examined. Nonlinear equations of transverse motion of the plate are expressed in terms of three displacement components. When the effects of transverse shear and rotatory inertia are neglected, these equations readily reduce to the displacement equations of the von Kármán plate theory. In order to study these effects on the dynamic behavior of circular plates, a solution for w is assumed in the polynomial form to satisfy the out-of-plane boundary conditions. The two inplane equilibrium equations are then solved exactly in conjunction with the required inplane boundary conditions. Using these inplane displacements and the transverse displacement assumed earlier, the equation of transverse motion is satisfied approximately by use of Galerkin's method. The resulting nonlinear ordinary differential equation for the time function is numerically integrated using the fourth-order Runge–Kutta procedure. Numerical results are presented for different plate parameters. Present results are in close agreement with existing solutions for all special cases[2, 3, 7].

2. GOVERNING EQUATIONS

The equations of transverse motion of a circular isotropic plate of radius a and thickness h (Fig. 1) including the effects of transverse shear deformation and rotatory inertia are[8]

$$u_{,xx}+\frac{1}{2}(1-\nu)u_{,yy}+\frac{1}{2}(1+\nu)v_{,xy}=-w_{,x}\left[w_{,xx}+\frac{1}{2}(1-\nu)w_{,yy}\right]-\frac{1}{2}(1+\nu)w_{,y}w_{,xy} \qquad (1)$$

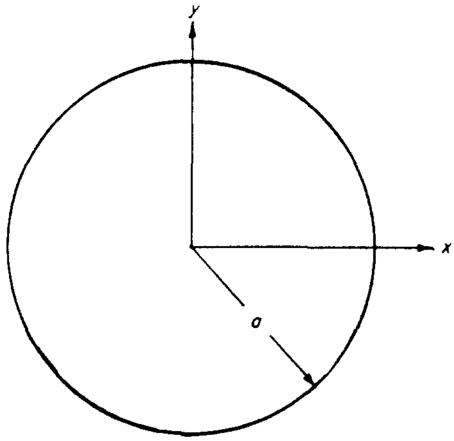

Fig. 1. Geometry and coordinate system of plate.

$$v_{,yy}+\frac{1}{2}(1-\nu)v_{,xx}+\frac{1}{2}(1+\nu)u_{,xy}=-w_{,y}\left[w_{,yy}+\frac{1}{2}(1-\nu)w_{,xx}\right]-\frac{1}{2}(1+\nu)w_{,x}w_{,xy} \tag{2}$$

$$L_1(I)+L_2(w)=0 \tag{3}$$

where u, v and w are three displacement components on the midsurface, x and y are the rectangular cartesian coordinates, a comma denotes the partial differentiation with respect to the corresponding coordinates and ν is the Poisson's ratio and where

$$I=q(x,y)-\rho h w_{,tt}+h(N_{xx}w_{,xx}+N_{yy}w_{,yy}+2N_{xy}w_{,xy}) \tag{4}$$

and the differential operators L_1 and L_2 are defined as

$$L_1=a_1\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}\right)+a_2\frac{\partial^4}{\partial x^2\partial y^2}+a_3\frac{\partial^4}{\partial t^4}+a_4\left(\frac{\partial^4}{\partial x^2\partial t^2}+\frac{\partial^4}{\partial y^2\partial t^2}\right)+a_5\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+a_6\frac{\partial^2}{\partial t^2}-1$$

$$L_2=a_7\left(\frac{\partial^4}{\partial x^2\partial t^2}+\frac{\partial^4}{\partial y^2\partial t^2}\right)+a_8\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}\right)+a_9\frac{\partial^4}{\partial x^2\partial y^2}+a_{10}\frac{\partial^6}{\partial x^2\partial y^2\partial t^2}+a_{11}\left(\frac{\partial^6}{\partial x^4\partial t^2}+\frac{\partial^6}{\partial y^4\partial t^2}\right)$$
$$+a_{12}\left(\frac{\partial^6}{\partial x^2\partial t^4}+\frac{\partial^6}{\partial y^2\partial t^4}\right)+a_{13}\left(\frac{\partial^6}{\partial x^6}+\frac{\partial^6}{\partial y^6}\right)+a_{14}\left(\frac{\partial^6}{\partial x^4\partial y^2}+\frac{\partial^6}{\partial y^4\partial x^2}\right). \tag{5}$$

In eqn (4) q is the lateral load per unit area of the plate, ρ is the mass density of the plate and N_{ij} are the inplane stress resultants per unit length given by

$$N_{xx}=C\left[u_{,x}+\frac{1}{2}w_{,x}^2+\nu\left(v_{,y}+\frac{1}{2}w_{,y}^2\right)\right]$$
$$N_{yy}=C\left[v_{,y}+\frac{1}{2}w_{,y}^2+\nu\left(u_{,x}+\frac{1}{2}w_{,x}^2\right)\right] \tag{6}$$
$$N_{xy}=\frac{1}{2}C(1-\nu)(u_{,y}+v_{,x}+w_{,x}w_{,y})$$

in which $C=Eh/(1-\nu^2)$ with E being Young's modulus of the plate material. The coefficients a_1–a_{14} in eqns (5) are:

$$a_1=-b_1b_2,\quad a_2=b_3^2-b_1^2-b_2^2,\quad a_3=-b_4^2,\quad a_4=b_4(b_1+b_2),$$
$$a_5=b_1+b_2,\quad a_6=-2b_4,\quad a_7=-\frac{R_i}{12},\quad a_8=\frac{1}{12(1-\nu^2)},\quad a_9=2a_8,$$
$$a_{10}=2a_7(b_3-b_1)+a_9b_4,\quad a_{11}=a_8b_4-a_7b_2,\quad a_{12}=a_7b_4,$$
$$a_{13}=-a_8b_2,\quad a_{14}=\left(a_8+\frac{a_9}{2}\right)(b_3+b_1)-\frac{a_9b_2}{2}$$

where

$$b_1 = \frac{T_s}{5(1-\nu)}, \quad b_2 = \frac{T_s}{10}, \quad b_3 = \frac{b_2(1+\nu)}{(1-\nu)}, \quad b_4 = 2b_2R_i(1+\nu).$$

In these coefficients the so-called tracing constants, T_s and R_i, are introduced to identify the terms which characterize the effects of the transverse shear deformation and rotatory inertia, respectively. If these effects are taken into consideration, then $T_s = R_i = 1$. The other three combinations are $T_s = 1$ and $R_i = 0$, $T_s = 0$ and $R_i = 1$, and $T_s = R_i = 0$. The last case leads to the dynamic von Kármán plate equations in three displacements.

The circular plate under consideration is assumed to be clamped along its boundary. The appropriate boundary conditions are, for a clamped stress-free plate,

$$\left.\begin{aligned} &w = w_{,x} = w_{,y} = 0 \\ &xN_{xx} + yN_{xy} = 0 \\ &yN_{yy} + xN_{xy} = 0 \end{aligned}\right\} \quad \text{along} \quad x^2 + y^2 = a^2 \tag{7}$$

and for a clamped immovable plate

$$w = w_{,x} = w_{,y} = u = v = 0 \quad \text{along} \quad x^2 + y^2 = a^2. \tag{8}$$

Equations (1)–(3) are to be solved in conjunction with the boundary conditions (7) or (8).

3. SOLUTION TO DYNAMIC EQUATIONS

A solution for w is sought in the separable form satisfying the required boundary conditions in eqns (7) and (8).

$$w = \frac{hF(\tau)}{a^4}(a^2 - x^2 - y^2)^2 \tag{9}$$

in which $F(\tau)$ is an unknown function of the nondimensional time $\tau = t\sqrt{(E/\rho a^2)}$. The two inplane displacements are chosen in the polynomial form as

$$u = \frac{F^2h^2}{a^8}[c_1x^7 + c_2x^5y^2 + c_3x^3y^4 + c_4xy^6 + a^2(c_5x^5 + c_6x^3y^2 + c_7xy^4) + a^4(c_8x^3 + c_9xy^2) + c_{10}a^6x] \tag{10}$$

$$v = \frac{F^2h^2}{a^8}[c_{11}y^7 + c_{12}y^5x^2 + c_{13}y^3x^4 + c_{14}yx^6 + a^2(c_{15}y^5 + c_{16}y^3x^2 + c_{17}yx^4) + a^4(c_{18}y^3 + c_{19}yx^2) + c_{20}a^6y]. \tag{11}$$

These expressions for u and v and expression (9) for w are substituted into eqns (1) and (2). By comparing the coefficients of like terms, twelve algebraic equations are generated in terms of the coefficients c_i. The additional eight equations required to determine c_1–c_{20} uniquely are now obtained by substituting eqns (9)–(11) in eqns (7) or (8). In each set of boundary conditions, eight and only eight algebraic equations are generated. The system of twenty nonhomogeous linear algebraic equations in coefficients c's are solved to give these coefficients explicitly. They are not defined here for the sake of brevity. The inplane displacements in eqns (10) and (11) are an exact solution to the inplane equilibrium eqns (1) and (2) for the assumed w given in eqn (9).

Generally, the deflection in eqn (9) and the inplane displacements in eqns (10) and (11) do not satisfy the dynamic equation of equilibrium in the transverse direction of the plate. In order to obtain an approximate solution to the nonlinear eqn (3), the Galerkin method is used here in this work. This procedure, after a lengthy calculation, yields the following differential equation for the time function $F(\tau)$:

$$k_1\frac{\mathrm{d}^6F}{\mathrm{d}\tau^6} + k_2\frac{\mathrm{d}^4}{\mathrm{d}\tau^4}(F + k_3F^3) + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(F + k_4F^3) + k_5F + k_6F^3 = \frac{q_0a^3}{3Eh^3k_7} \tag{12}$$

where q_0 is the intensity of uniformly distributed lateral load and coefficients k_i are obtained following a very lengthy standard mathematical procedure for the stress-free and immovable cases. Equation (12) includes the effects of transverse shear deformation and rotatory inertia. If either of these or both are zero, i.e. $T_s = 1$ and $R_i = 0$, $T_s = 0$ and $R_i = 1$, and $T_s = R_i = 0$, then the time-differential equation reduces to the Duffing-type equation as

$$\frac{d^2F}{d\tau^2} + k_5F + k_6F^3 = \frac{q_0a^3}{3Eh^3k_7}. \quad (13)$$

The time-differential eqn (12) is numerically integrated using the Runge–Kutta method whereas eqn (13) for $q_0 = 0$ can be solved exactly by the elliptic-integral method[1]. For static large-deflection problems F is independent of time and, hence, eqn (13) will reduce to a nonlinear algebraic lead-deflection relation in the large-deflection regime.

4. NUMERICAL RESULTS, DISCUSSION AND CONCLUSION

Numerical results are presented for static and dynamic behavior of an isotropic circular plate in Figs. 2–6. The value of ν is taken to be 0.3. The ratio of the nonlinear period T of vibration, including effects of transverse shear and rotatory inertia, to the corresponding linear period T of a classical plate, excluding these effects, was computed for different nondimensional amplitudes, radius-to-thickness ratios and boundary conditions. For the purpose of comparison, numerical results are also presented for the case when the transverse shear and rotatory inertia effects are not considered. In using the fourth-order Runge–Kutta procedure for the solution of eqn (12), the nondimensional time interval $\nabla\tau$ was taken as 0.0001.

When the effects of transverse shear and rotatory inertia are not taken into account, the present fundamental linear frequencies agree very well with those given in [2, 3]. On account of these effects, a comparison of present results with those of Ref. [7] for a clamped immovable plate is presented in Table 1. Good agreement is noted.

In the static case when the effect of transverse shear is ignored, the variation of the central deflection w_0 with the lateral uniform pressure q_0 is shown in Fig. 2 for a circular plate. It is observed that the present results are in excellent agreement with those of Yamaki[2] and Nowinski[3] for the immovable and stress-free cases, respectively. In Figs. 3 and 4 the period ratio T/T_0 is plotted against the nondimensional amplitude of vibration for different radius-to-thickness ratios of a circular plate. Figure 3 shows the results for the stress-free case whereas the results in Fig. 4 are for the immovable case. In these figures the effects of transverse shear and rotatory inertia are included except for the curves for which $T_s = R_i = 0$. For all the boundary conditions considered here, the period ratio decreases with increasing the amplitude of vibration, thereby exhibiting the hardening type of nonlinearity. The transverse shear and rotatory inertia effects increase the period ratio at any amplitude of vibration. When the

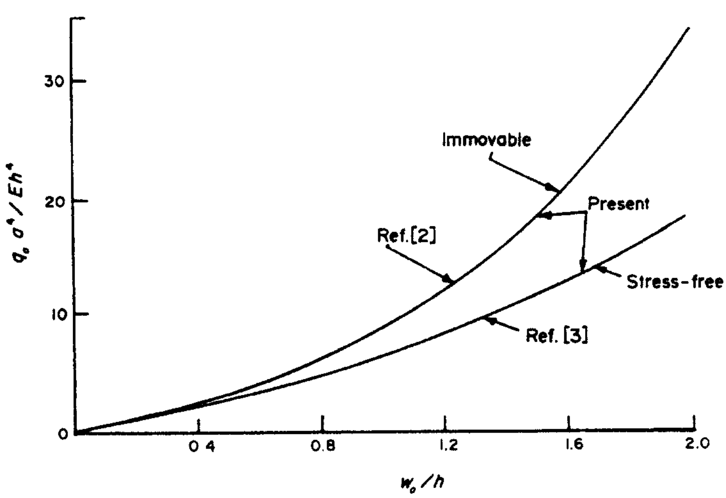

Fig. 2. Load-deflection curves for circular plate under uniformly distributed load.

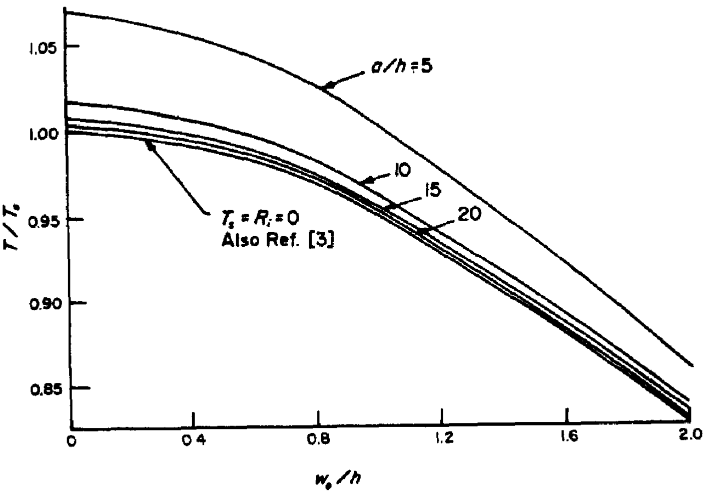

Fig. 3. Amplitude-period response curves for stress-free circular plate with different radius-to-thickness ratios.

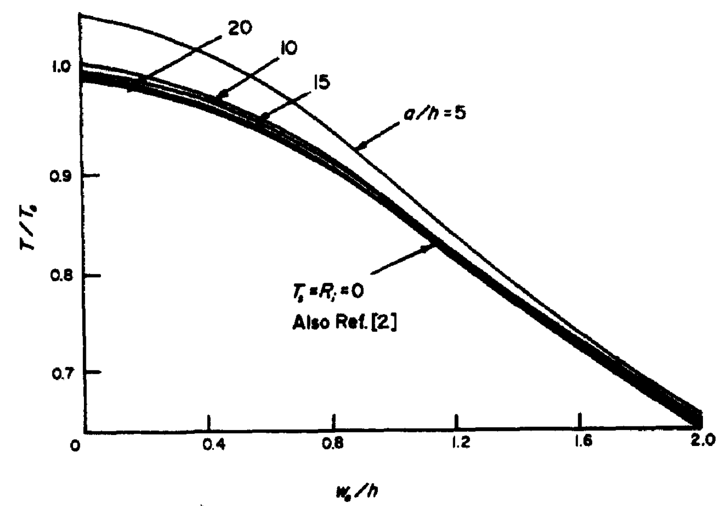

Fig. 4. Relation between period ratio and amplitude for immovable circular plate with various values of radius-to-thickness ratio.

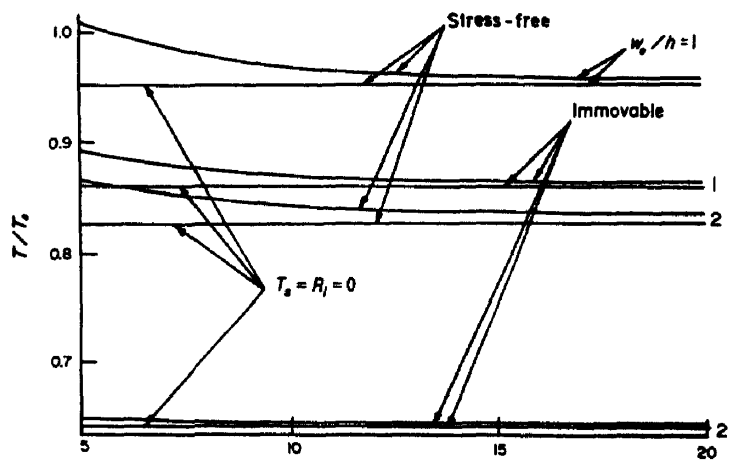

Fig. 5. Variation of period ratio with radius-to-thickness ratio for different nondimensional amplitudes.

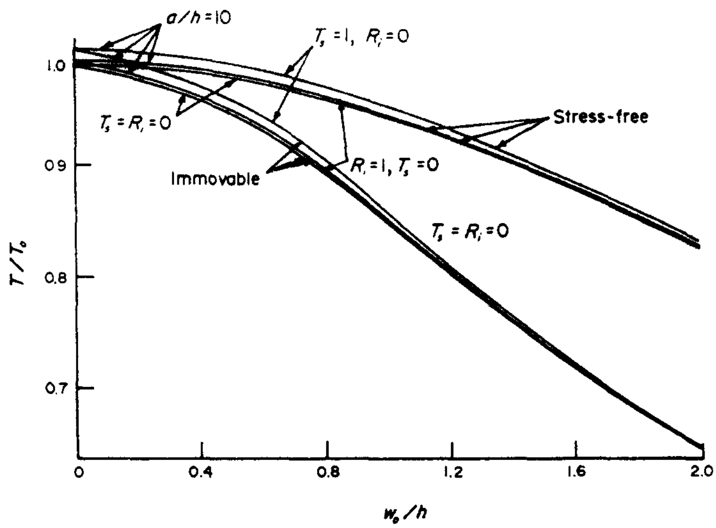

Fig. 6. Individual effect of transverse shear and rotatory inertia on amplitude-period response for stress-free and immovable circular plates.

amplitude increases, the stretching of the plate midsurface increases with a corresponding decrease in the bending and transverse shear effects. In view of this, the effect of transverse shear decreases with the amplitude. When the effects of transverse shear and rotatory inertia are both neglected ($T_s = R_i = 0$), present results are in excellent agreement with those of Refs. [2, 3].

The effect of the radius-to-thickness ratio on the large amplitude vibration of a circular plate at the amplitude equal to and twice the plate thickness is shown in Fig. 5. The horizontal lines represent the case when the effects of transverse shear and rotatory inertia are neglected. In this case, the period ratio is independent of the radius-to-thickness ratio. However, these effects are important for moderately thick plates particularly with stress-free boundary conditions. As the ratio a/h increases, the influences of transverse shear and rotatory inertia decrease. Each curve, therefore, approaches asymptotically the one for the corresponding classical thin plate. The individual effect of transverse shear ($T_s = 1$, $R_i = 0$) and rotatory inertia ($R_i = 1$, $T_s = 0$) on the nonlinear vibration behavior is shown in Fig. 6. It can be seen that the effect of transverse shear is more significant than that of rotatory inertia. Consideration of either the transverse shear effect or the rotatory inertia effect leads to an incrêase in the period ratio at any amplitude of vibration.

In conclusion, it is to be restated that in this study the effects of transverse shear and rotatory inertia are incorporated into the dynamic von Kármán nonlinear plate equations in such a manner that these effects on the dynamic behavior of circular plates can be investigated individually or totally. These effects decrease with increasing the amplitude of vibration and are maximum at infinitesimal small amplitudes. The hardening effect is considerably less for a

Table 1. Values of period ratio for clamped immovable circular plate

$\frac{w_o}{h}$	a/h = 5		a/h = 10		a/h = 20		Classical Thin Plate	
	Present	[7]	Present	[7]	Present	[7]	Present	[7]
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	0.9911	0.9921	0.9923	0.9927	0.9925	0.9928	0.9928	0.9928
0.4	0.9669	0.9699	0.9711	0.9718	0.9723	0.9722	0.9724	0.9724
0.6	0.9303	0.9366	0.9390	0.9402	0.9412	0.9410	0.9413	0.9413
0.8	0.8855	0.8965	0.8988	0.9015	0.9024	0.9026	0.9026	0.9029
1.0	0.8369	0.8533	0.8544	0.8591	0.8591	0.8603	0.8597	0.8607

stress-free edge than for an immovable edge. For circular plates with stress-free boundaries, the effects of transverse shear and rotatory inertia are of considerable significance even at large amplitudes of vibration.

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